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SIMPLIFIED THEORY FOR THE BEHAVIOUR OF BURIED FLEXIBLE CYLINDERS UNDER THE INFLUENCE OF UNIFORM HOOP COMPRESSION

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Abstract—The elastic behaviour of a long flexible cylinder buried in an infinite elastic medium and acted on by uniform hoop compressions is examined. A simplified solution is obtained, which can be used to accurately and efficiently determine the cylinder response.

The simplified theory is used to obtain a parametric solution to the cylinder buckling problem. The effects of interface condition, elastic ground parameters, load behaviour, and finite thickness on the critical hoop compression are considered.

A straightforward method is developed for predicting the prebuckling displacements and bending moments which result from the influence of initial geometrical imperfections and nonhydrostatic field stresses.

1. INTRODUCTION

When a circular cylindrical tube is buried in a soil or rock mass, the pre-existing field stresses in the ground will induce hoop compressions in the cylinder. These hoop forces reduce the cylinder stiffness, and elastic instability may result, or the nonuniformities present in the system may cause significant displacements which damage the cylinder.

This study considers the problem of a long flexible cylindrical tube buried in an infinite elastic medium which is initially in a state of hydrostatic compression. Other authors have examined the elastic stability of these structures[1–6] on the basis that the hoop compressions for many such structures are approximately uniformly distributed[7, 8]. The present work begins by examining the application of the shell theories of Flugge[9] and Herrmann and Armenakas[10] to this problem, and the linearised equations of equilibrium are formulated. Linear elastic continuum theory is used to determine the ground response, and a comparison with the more complex model used by Forrestal and Herrmann[3] justifies the simpler approach.

An inextensional shell theory is then used to obtain a remarkably simple and accurate solution. This facilitates a parametric study of the buckling problem, where the effects of finite cylinder thickness, load behaviour, elastic ground parameters, and interface behaviour are considered. In addition, the simplified solution is used to estimate the prebuckling displacements and stress resultants associated with any nonuniformities present in the system. Nonuniformities in both initial stresses and initial geometry are examined.

2. STATEMENT OF PROBLEM

The buried tube is assumed to be very long, so that it deforms under conditions of plane strain. The tube or "tunnel lining" is assumed to behave elastically with Young's modulus E_l and Poisson's ratio ν_l. The uniform thickness t is assumed to be small relative to the average radius of the midsurface of the cylinder a, say $t/a < 0.05$, so that ring theory may be used to model the structural behaviour. The analysis of thicker tubes requires a more elaborate theory (see, for example, Renton[11]) and because these thicker structures usually fail inelastically, they will not be considered.

The ground that supports the tube is considered to be a single phase isotropic material such as a soil or rock mass, with an incrementally elastic behaviour characterised by two constants: Young's modulus E_s and Poisson's ratio ν_s.

Before insertion of the tube, the ground is assumed to be prestressed with a uniform hydrostatic stress p, which induces a uniform compressive hoop force N in the tube.

Two alternative conditions will be assumed to characterise the soil-structure interaction response at the interface.

(a) A perfectly rough interface across which there is complete compatability of radial and circumferential dispacements, and full transmission of normal and shear tractions.

(b) A perfectly smooth interface, which does not transmit shear stresses between the structure and ground, and across which circumferential displacements are not continuous, due to interfacial slip.

The real interface conditions will, of course, be somewhere between these two extremes, because there will, in general, be some finite limit to the shear stresses that can be developed between the cylinder and ground. These two ideal conditions, however, provide useful bounds on the system response.

It is not feasible to model the precise way in which real ground applies load to a structure at the soil–surface interface. During the buckling or prebuckling deformations, the loads applied to the tube may well rotate, see Fig. 2. The exact nature of this "load behaviour" depends on the response of the ground material, whether it be rock or soil, cohesive or cohesionless, and is not fully understood. Now, it is well known that the stability of unsupported rings is significantly influenced by this type of load behaviour[12–14]. In order to gain some insight into this aspect of buried cylinder stability, then, the tractions transmitted to the cylindrical tube from the ground will be assumed to exhibit three alternative types of ideal behaviour:

(a) Constant directional behaviour—where the applied tractions remain constant in direction with respect to the initial tube geometry.

(b) Hydrostatic behaviour—where the applied tractions remain normal to the deforming tube surface.

(c) Centre-directed behaviour—where the tractions remain directed towards the cylinder axis as their points of application move during cylinder deformation.

For the sake of simplicity, it will be assumed that the initial stress effect can be represented by these three alternate load behaviours. Forrestal and Herrmann[3] adopted a different approach where they examined the reorientation of the initial stresses in the continuum and their influence on the incremental tractions at the interface, for a perfectly elastic continuum. It will be shown subsequently that the solution of Forrestal and Herrmann[3] is significantly different to the simpler ones presented in this study, only when the ground is soft and perfectly elastic under conditions of high initial stress, and that, in general, the load behaviour does not significantly affect the cylinder stability where the ground support is significant.

For the examination of prebuckling displacements, the imperfections in geometry and nonuniformities of initial stress will be assumed to be small, so that the hoop compression in the tube remains uniform. The present theory is based on a linearised shell theory, so that effect of prebuckling displacements on the elastic stability is neglected. An imperfect tube may never, in fact, buckle, and nonlinear theory is required if this aspect of the problem is to be examined (e.g. [15]). For the present study, the word *prebuckling* will refer to the region where loads are less than those that cause a perfectly circular tube to buckle. This definition is employed because it is consistent with the linear theory, and it is further supported by nonlinear analysis which indicates that imperfect buried flexible tubes do, in fact, buckle[15].

3. EXTENSIONAL SOLUTION

Shell response

In their solution to the buried cylinder problem, Forrestal and Herrmann[3] made use of the rigorous shell theory developed by Herrmann and Armenakas[10] for the study of the effect of initial stresses on the static and dynamic behaviour of cylindrical shells. For the present problem, where the initial hoop compressions are assumed to be uniform, the shell theory reduces to the well-known theory of Flugge[9]. Use of this Flugge–Herrmann theory to describe the behaviour of a "ring" (or uniform cylindrical shell acting under conditions of plane strain) with flexural stiffness

$$D = \frac{E_l t^3}{12(1 - \nu_l^2)} \tag{1a}$$

and hoop stiffness

$$H = \frac{E_l t}{(1 - \nu_l^2)} \tag{1b}$$

leads to expressions for the hoop force N and bending moment M in terms of the radial and circumferential displacements of the ring midsurface w and v respectively, see Fig. 1,

$$\begin{aligned} N &= \frac{H}{a}\left(\frac{\partial v}{\partial \theta} + w\right) + \frac{D}{a^3}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right) \\ M &= -\frac{D}{a^2}\left(w + \frac{\partial^2 w}{\partial \theta^2}\right), \end{aligned} \tag{2}$$

where θ is the circumferential coordinate.

Substitution of these expressions into the linearised differential equations of equilibrium of the Flugge-Herrmann theory results in expressions for the radial F_r and circumferential F_θ tractions at the soil-surface interface. Using the Fourier harmonic decomposition

$$\begin{aligned} (w, F_r) &= \sum_{n=2}^{\infty} (W_n, \sigma_n) \cos n\theta \\ (v, F_\theta) &= \sum_{n=2}^{\infty} (V_n, \tau_n) \sin n\theta \end{aligned} \tag{3}$$

those expressions

$$\left[A_n - \frac{N}{a^2} B_n\right] \begin{bmatrix} V_n \\ W_n \end{bmatrix} = \begin{bmatrix} \tau_n \\ \sigma_n + \dfrac{nt}{2a} \quad \tau_n \end{bmatrix} \tag{4}$$

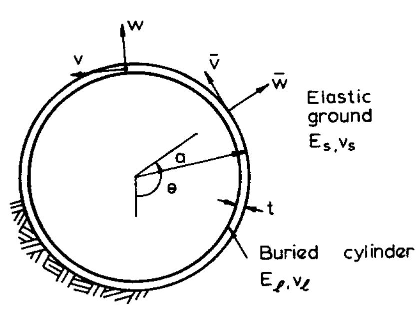

Fig. 1. Geometry of buried cylinder.

for the "nth" Fourier harmonic, where

$$A_n = \frac{H}{a^2}\begin{bmatrix} n^2 & n \\ n & 1 + \dfrac{D(n^2 - 1)}{Ha^2} \end{bmatrix}$$
$$B_n = \begin{bmatrix} n^2 + 1 & 2n \\ 2n & n^2 + 1 \end{bmatrix}. \tag{5}$$

The term $(nt/2a)\tau_n$, which appears in the right-hand side of eqn (4), is associated with the bending moment about the ring midsurface resulting from the shear traction at the external face of the ring.

In previous studies of this problem, the buckling strength of the buried cylinder has been presented as a critical pressure p_b acting on the cylinder. Examination of the static interaction of the cylinder–ground system indicates that this stress acting on the structure at the interface is some function of, but not necessarily equal to, the hydrostatic field stress within the ground,[16, 17] (pa is the maximum value of N, which decreases as the ground stiffness increases). In order to make this distinction clear, the hoop compression N is used to represent the destabilising stresses in the cylinder.

Ground response

It is now necessary to develop a stiffness relation between the coefficients of ground displacement and tractions applied to the ground at the interface.

In the solution of Forrestal and Herrmann[3], the destabilising effect of initial stresses on the stiffness of the ideal elastic continuum surrounding the tube were considered. Nonlinear differential equations of equilibrium were presented which, when linearised with respect to the initial hydrostatic field stress p, have the same form as the static equations, but with modified elastic constants

$$\lambda_s = \lambda^* + p$$
$$G_s = G^* + p, \tag{6}$$

where λ^* and G^* are, respectively, Lame's constant and the shear modulus of the unstressed ground. Just as these terms were "neglected" by Forrestal and Herrmann, in the present work the response of the elastic continuum will be determined using the well-known solution to the static equations of equilibrium in conjunction with elastic parameters λ_s and G_s measured relative to the continuum in its prestressed state (the usual geotechnical practice), so that this initial stress effect is included implicitly.

Using linear elastic theory (e.g. Timoshenko and Goodier[18]) the coefficients of radial displacement $\overline{w}$, circumferential displacement $\overline{v}$, radial traction $\overline{\sigma}$, and circumferential traction $\overline{\tau}$ are related by

$$-\frac{G_s}{a}\begin{bmatrix} n + 1 + \dfrac{n-1}{\chi_s} & n + 1 - \dfrac{n-1}{\chi_s} \\ n + 1 - \dfrac{n-1}{\chi_s} & n + 1 + \dfrac{n-1}{\chi_s} \end{bmatrix}\begin{bmatrix} \overline{V}_n \\ \overline{W}_n \end{bmatrix} = \begin{bmatrix} \overline{\tau}_n \\ \overline{\sigma}_n \end{bmatrix}, \tag{7}$$

where $\chi_s = 3 - 4\nu_s$, the shear modulus $G_s = E_s/2(1 + \nu_s)$, and the Fourier harmonic decomposition used is

$$(\overline{w}, \overline{\sigma}) = \sum_{n=2}^{\infty} (\overline{W}_n, \overline{\sigma}_n)\cos n\theta$$
$$(\overline{v}, \overline{\tau}) = \sum_{n=2}^{\infty} (\overline{V}_n, \overline{\tau}_n)\sin n\theta. \tag{8}$$

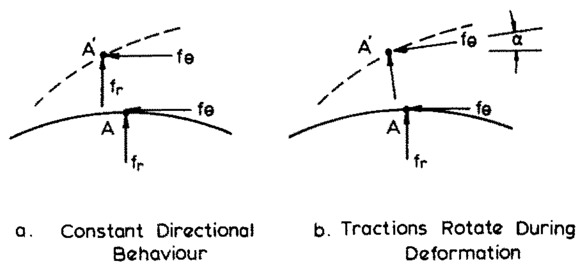

Fig. 2. Behaviour of interface tractions during deformation.

Interaction at the interface

In order to specify the soil–structure interaction behaviour at the interface, the composition of the tractions on the lining F_r and F_θ needs to be examined. Each harmonic of these tractions consists of three components

$$\begin{aligned} \sigma_n &= \sigma_n^e + \sigma_n^i + \sigma_n^a \\ \tau_n &= \tau_n^e + \tau_n^i + \tau_n^a, \end{aligned} \tag{9}$$

where σ^i and τ^i are the incremental tractions which result from any rotation of the initial tractions during the deformation; σ^a and τ^a are the tractions which represent the ground support at the interface; and σ^e and τ^e are any other tractions applied to the interface.

The tractions σ^e and τ^e represent any additional tractions applied to the perfectly circular cylinder in its uniformly prestressed state. They are used to introduce the effects of nonuniformities such as initial geometrical imperfections, nonhydrostatic initial stress or nonuniform initial stress. For the classical stability problem they are identically zero.

The ground influences the structure through the initial stress terms σ^i and τ^i, and the ground support terms σ^a and τ^a. As the structure deforms, the uniform radial traction N/a may rotate through an angle α relative to its initial direction, see Fig. 2, producing an additional traction in the circumferential direction, $\tau^i = N\alpha/a$. Table 1 shows the angles of rotation for the three ideal load behaviours. The deformation at the interface is related to the midsurface displacements by

$$\begin{aligned} v^i &= v + \frac{t}{2a}\left(v - \frac{\partial w}{\partial \theta}\right) \\ w^i &= w, \end{aligned} \tag{10}$$

and this leads to expressions for the coefficients of radial traction σ_n^i and circumferential traction τ_n^i

$$\begin{bmatrix} \tau_n^i \\ \sigma_n^i \end{bmatrix} = \frac{N}{a^2} \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} V_n \\ W_n \end{bmatrix}, \tag{11}$$

where the values of the coefficients l_{ij} are given in Table 2.

The interface condition directly effects the way in which the ground supports the structure. Table 3 gives details of the soil structure interaction at the interface corresponding to the rough and smooth interface conditions. These relations are used in

Table 1. Rotation of initial traction

Angle of rotation	Constant directional	Hydrostatic	Centre directed
α	0	$\frac{1}{a}\left(\frac{\partial w^i}{\partial \theta} - v^i\right)$	$-\frac{v^i}{a}$

Table 2. Load behavior matrix

Constant directional	$l_{11} = 0$ $l_{21} = 0$	$l_{12} = 0$ $l_{22} = 0$
Hydrostatic	$l_{11} = -1$ $l_{21} = 0$	$l_{12} = -n$ $l_{22} = 0$
Centre-directed	$l_{11} = -1$ $l_{21} = 0$	$l_{12} = -\frac{nt}{2a}$ $l_{22} = 0$

Table 3. Conditions at the interface

	Rough interface	Smooth interface
Compatability of displacements	$W_n = \overline{W}_n$ $V_n + W_n \frac{nt}{2a} = \overline{V}_n$	$W_n = \overline{W}_n$ $V_n \neq \overline{V}_n$
Equilibrium of stresses	$\sigma_n^a = \bar{\sigma}_n$ $\tau_n^a = \bar{\tau}_n$	$\sigma_n^a = \bar{\sigma}_n$ $\tau_n^a = \bar{\tau}_n = 0$

conjunction with eqn (7) to obtain expressions for the coefficients of σ^a and τ^a

$$\begin{bmatrix} \tau_n^a \\ \sigma_n^a \end{bmatrix} = - \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} V_n \\ W_n \end{bmatrix}, \tag{12}$$

where the values of the coefficients m_{ij} are given in Table 4.

The problem can now be solved using the various components which have been defined. Using the Flugge–Herrmann shell theory (4) and (5) in conjunction with eqns (9), (11), and (12) leads to the expressions

$$\left[A_n^* - \frac{N}{a^2} B_n^* \right] \begin{bmatrix} V_n \\ W_n \end{bmatrix} = \begin{bmatrix} \tau_n^e \\ \sigma_n^e \end{bmatrix}, \tag{13}$$

where

$$\begin{aligned} A_n^* &= A_n + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} + \frac{nt}{2a} m_{11} & m_{22} + \frac{nt}{2a} m_{12} \end{bmatrix} \\ B_n^* &= B_n + \begin{bmatrix} l_{11} & l_{12} \\ l_{21} + \frac{nt}{2a} l_{11} & l_{22} + \frac{nt}{2a} l_{12} \end{bmatrix} \end{aligned} \tag{14}$$

and the values of l_{ij} and m_{ij} are provided in Tables 2 and 4, respectively.

Table 4. Ground support matrix.

Rough	$m_{11} = \frac{2G_s[2n(1 - \nu_s) + (1 - 2\nu_s)]}{a \quad (3 - 4\nu_s)}$	$m_{12} = \frac{G_s}{a} [(n + 1)(1 + nt/2a) - \frac{(n - 1)}{(3 - 4\nu_s)} (1 - nt/2a)]$
	$m_{21} = \frac{2G_s[n(1 - 2\nu_s) + 2(1 - \nu_s)]}{a \quad (3 - 4\nu_s)}$	$m_{22} = \frac{G_s}{a} [(n + 1)(1 + nt/2a) + \frac{(n - 1)}{(3 - 4\nu_s)} (1 - nt/2a)]$
Smooth	$m_{11} = 0$ $m_{21} = 0$	$m_{12} = 0$ $m_{22} = \frac{2G_s}{a} \frac{n^2 - 1}{2n(1 - \nu_s) + (1 - 2\nu_s)}$

4. INEXTENSIONAL SOLUTION

Thin rings ($t/a \ll 1$) have hoop stiffnesses relatively high compared to their flexural stiffnesses and, as a result, deform in bending with little associated extension of the mid-surface. It would, thus, seem reasonable to neglect this extension when determining the response of buried cylinders, that is take

$$\epsilon_{\theta\theta} = \frac{1}{a}\left(\frac{\partial v}{\partial \theta} + w\right) = 0. \tag{15}$$

This assumption leads to a relationship between circumferential and radial displacement coefficients for the ring

$$V_n = -\frac{W_n}{n}, \tag{16}$$

which is used to simplify the extensional formulation.

The Flugge–Herrmann shell theory can be modified using eqn (15), so that a single equilibrium equation is developed. Substitution of (3) then yields an expression for the coefficient of radial displacement of the shell mid-surface.

$$W_n = \frac{\sigma_n^e - \tau_n^e/n}{\dfrac{D}{a^4}(n^2 - 1)^2 - \dfrac{N}{a^2}\left(\dfrac{(n^2 - 1)^2}{n^2} + L_n\right) + M_n}, \tag{17}$$

where L_n and M_n represent the effects of load behaviour and ground restraint, respectively, and expressions for them are given in Table 5. After evaluation of W_n using eqn (17), V_n can be found from (16).

The tabulated values of L_n and M_n are in two groups. The first column represents values obtained directly from the extensional theory after the application of eqn (16). If all terms in t/a are neglected, the simpler expressions appearing in the second column are obtained.

Instability will occur when the denominator in (17) is zero, so that

$$\frac{N}{a^2} = \frac{\dfrac{D}{a^4}(n^2 - 1)^2 + M_n}{\dfrac{(n^2 - 1)^2}{n^2} + L_n}. \tag{18}$$

Table 5. Factors for load behaviour and soil restraint—Inextensional theory.

Symbol	Description of condition	(1)	(2)
L_n for load behavior	Constant directional pressure	0	0
	Hydrostatic pressure	$\left(\frac{n^2-1}{n^2}\right)\left[1 - n\left(\frac{nt}{2a}\right)\right]$	$\frac{(n^2-1)}{n^2}$
	Centre-directed pressure	$-\frac{1}{n^2}\left[1 - 2n\left(\frac{nt}{2a}\right) + n^2\left(\frac{nt}{2a}\right)^2\right]$	$-\frac{1}{n^2}$
M_n for soil restraint	Rough interface	$\frac{G_s}{a}\frac{(n^2-1)}{n^2}\left[n - 1 + \frac{n+1}{\chi_s}\right] + \frac{G_s}{a}2\left(\frac{nt}{2a}\right)\frac{n^2-1}{n}\frac{\chi_s - 1}{\chi_s} + \frac{G_s}{a}\left(\frac{nt}{2a}\right)^2\left[n + 1 + \frac{n-1}{\chi_s}\right]$	$\frac{G_s}{a}\frac{n^2-1}{n^2}\left[n - 1 + \frac{n+1}{\chi_s}\right]$
	Smooth interface	$\frac{G_s}{a}\frac{4(n^2-1)}{[n - 1 + \chi_s(n+1)]}$	$\frac{G_s}{a}\frac{4(n^2-1)}{[n - 1 + \chi_s(n+1)]}$

The critical hoop compression N_{cr} is the lowest such value of N, and the critical mode is the corresponding harmonic n_{cr}.

For hydrostatic load behaviour, rejecting any additional terms in t/a, the expression (18) takes the form:

$$\frac{N}{a^2} = \frac{D}{a^4}(n^2 - 1) + \frac{2G_s}{a}\frac{2n(1 - \nu_s) - (1 - 2\nu_s)}{n^2(3 - 4\nu_s)} \tag{19a}$$

when the interface is rough and

$$\frac{N}{a^2} = \frac{D}{a^4}(n^2 - 1) + \frac{2G_s}{a}\frac{1}{2n(1 - \nu_s) + (1 - 2\nu_s)} \tag{19b}$$

when the interface is smooth. Forrestal and Herrmann[3] obtained similar expressions by simplifying the explicit expressions from their extensional solution. The rough solutions differ somewhat, as a result of the additional initial stress terms Forrestal and Herrmann[3] employed in determining the ground response, but the simplest expressions for the smooth case are equivalent.

The simplified expressions still need to be examined for a range of modes n, to determine the critical (lowest) value of hoop compression N_{cr}. When n is large, it proves to be effective to approximate the discrete variable n by a continuous variable $\bar{n}$, and to minimise the expressions (19) analytically. For a rough interface it follows that the critical value of $\bar{n}$

$$\bar{n}_{cr} = \left[\frac{2G_s a^3}{D}\frac{(1 - \nu_s)}{(3 - 4\nu_s)}\right]^{1/3}, \tag{20a}$$

and for a smooth interface

$$\bar{n}_{cr} = \left[\frac{2G_s a^3}{D}\frac{1}{4(1 - \nu_s)}\right]^{1/3} \tag{20b}$$

with

$$\frac{N_{cr}}{a^2} \simeq \frac{3D\bar{n}_{cr}^2}{a^4}. \tag{20c}$$

For the special case of incompressible ground, $\nu_s = \frac{1}{2}$, as is the case for clays under undrained conditions,

$$\bar{n}_{cr} = \left[\frac{G_s a^3}{D}\right]^{1/3} \tag{21}$$

for both interface conditions. It is apparent from eqns (20, 21) that as the relative flexural stiffness $D/G_s a^3$ of the cylinder decreases, the approximate critical harmonic $\bar{n}_{cr}$ increases.

A number of different solutions to the buried cylinder stability problem have been developed, and these have been examined by the authors in a recent work[19]. A comparison of the extensional and inextensional theories indicated that the buckling deformations are very nearly inextensional, and that the simpler theory can be used with complete confidence. The differences between formulations based on the Flugge–Herrmann shell theory and the Donnell shell theory were also examined, and it was found that either approach leads to a satisfactory solution of the buried tube problem.

5. PARAMETRIC STUDY

Introduction

The simple theory which has been developed permits a parametric study of the buried cylinder problem. This study facilitates the use of the solution, and leads to a greater understanding of the behaviour of buried cylinders.

Consider a long flexible cylinder, buried at great distance from the surface, under the influence of pre-existing hydrostatic field stress. A uniform hoop compression N is generated in the cylindrical shell, which may cause elastic instability in the system. The critical value of hoop compression can be expressed as

$$N_{cr} = \frac{3D}{a^2}\left[\left(\frac{G_s a^3}{D}\right)^{2/3} + 1\right] IR_l R_t, \tag{22}$$

where D is the flexural rigidity of the cylindrical shell under plane strain conditions; a is the radius of the cylindrical shell; G_s is the shear modulus of the ground; I is the influence factor dependent on the interface conditions and Poisson's ratio of the soil, Fig. 3; R_l is the correction factor for the effect of load behaviour, Fig. 4; and R_t is the correction factor for the effect of the shell thickness, Fig. 5.

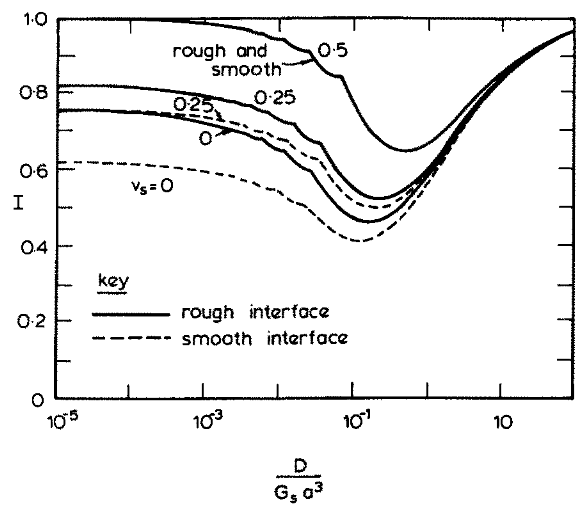

Fig. 3. Influence factor for elastic stability.

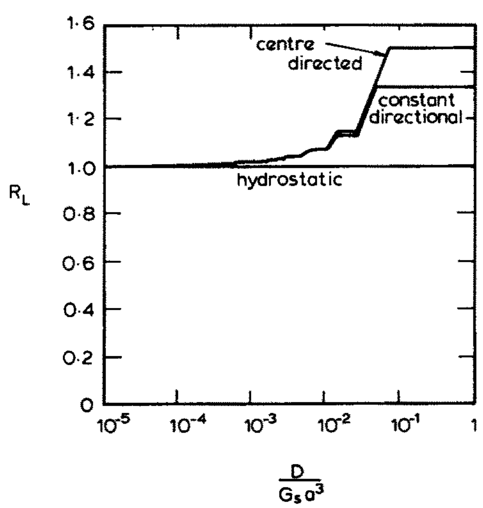

Fig. 4. Load behaviour correction factor R_L.

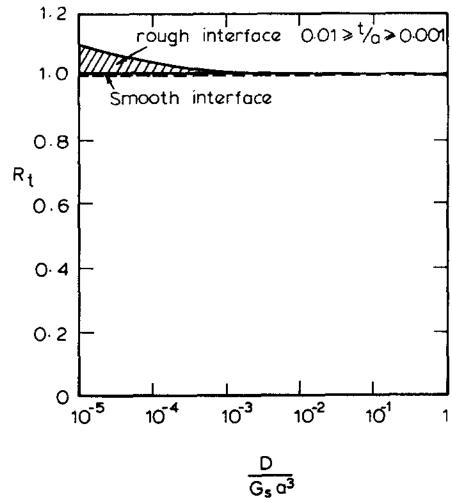

Fig. 5. Cylinder thickness correction factor R_t.

The parametric equation (22) has been chosen on the basis of eqns (20c) and (21), in conjunction with the critical hoop compression for an unsupported cylinder $N_{cr} = 3D/a^2$.

Effect of ground parameters

The most significant factor affecting the stability of any particular buried cylinder is the relative stiffness of the ground in which it is embedded. Each particular problem has been characterised in this parametric solution by:

(i) The flexural stiffness of the cylinder relative to the shear modulus of the ground $D/G_s a^3$. This directly influences the critical harmonic n_{cr}, as well as the load behaviour and thickness effects, see Figs. 4 and 5.

(ii) Poisson's ratio of the ground ν_s, which also affects the critical hoop compression directly, see Fig. 3.

The importance of these two parameters will be discussed further in the following sections.

Effect of interface condition

Examination of Figs. 3 and 5 leads to the following observations concerning the influence of interface condition:

(i) In general, rough cylinders are more stable than those which are perfectly smooth, with a maximum difference of about 15% when the ground is relatively stiff.

(ii) Poisson's ratio of the ground ν_s significantly affects the influence of interface condition, and for the special case of incompressible ground, $\nu_s = \frac{1}{2}$, there is no difference between rough and smooth cylinder response.

(iii) The finite thickness of the tube is significant only if the cylinder is rough, so that for smooth cylinders $R_t = 1$.

Effect of load behaviour

Three alternative types of load behaviour have been examined. From Fig. 4 it can be seen that

(i) The correction factor R_l can be omitted for a conservative solution, since $R_l \geq 1$.

(ii) The effect of load behaviour is negligible if the ground is relatively stiff such that $D/G_s a^3 < 10^{-3}$ and $n_{cr} \geq 6$.

(iii) Centre-directed behaviour is most stable, and hydrostatic the least stable.

Effect of cylinder thickness

Since tractions are applied to the buried tube at the external face, they can induce bending moments about the midsurface of the ring. This finite thickness effect, Fig. 5,

(i) is proportional to the relative thickness t/a as well as the critical mode of deformation n_{cr},
(ii) can be significant when the ground is relatively stiff and the critical mode n_{cr} is high, and
(iii) always acts to increase the stability of the tube and can be neglected to yield a conservative solution

6. PREBUCKLING DISPLACEMENTS

The behaviour of a cylindrical shell inserted into an infinite elastic continuum has been examined, and the hoop compressions which cause elastic instability have been determined. A simple method of calculating the prebuckling displacements and stress resultants will now be developed, which is valid when the displacements are small and the initial hoop compression in the tube is approximately uniform.

Various factors affecting buried cylinders introduce nonuniformities, which cause a cylinder to deform prior to buckling. The equations which have been developed are for perfectly circular cylindrical shells under conditions of uniform hydrostatic initial stress. Initial stresses which are slightly nonhydrostatic result in non-zero tractions σ^e and τ^e, which cause deformation. Similarly, the nonuniform curvature associated with noncircular initial shape can be represented by non-zero tractions σ^e and τ^e applied to a perfectly circular ring[19]. Details of these non-zero values of σ^e and τ^e are provided in Table 6.

If the tractions σ^e and τ^e have harmonic coefficients σ_n^e and τ_n^e, then the deformation of the tube may be calculated using eqn (17). For the case where the load behaviour is hydrostatic,

$$W_n = \frac{(\sigma_n^e - \tau_n^e/n)}{\left(\frac{D}{a^4}(n^2-1)^2 - \frac{N}{a^2}(n^2-1) + M_n\right)}, \tag{23}$$

and this expression can be used in conjunction with the condition of zero extension

$$V_n = -W_n/n \tag{24}$$

to determine the response of the tube.

The stress resultants are found using eqn (2), after substitution of the harmonic components of w

$$\begin{aligned} M &= \sum_{n=2}^{\infty} \frac{D}{a^2}(n^2-1)W_n \cos n\theta \\ N &= \sum_{n=2}^{\infty} \frac{D}{a^2}(1-n^2)W_n \cos n\theta. \end{aligned} \tag{25}$$

Table 6. Nonuniformities causing prebuckling displacement.

Description		
Uniform nonhydrostatic pressure	Coefficient of lateral pressure K	$\sigma_2^e = \frac{N}{a}(1-K)/2$ $\tau_2^e = \frac{N}{a}(K-1)/2$ $\sigma_n^e = \tau_n^e = 0$ for $n \neq 2$
Imperfect circular shape	Radial imperfection $\epsilon(\theta)$ Radius $= a + \epsilon$	$\epsilon_n = \frac{1}{\pi}\int_0^{2\pi} \epsilon \cos n\theta d\theta$ $\sigma_n^e = \frac{N}{a}\frac{\epsilon_n}{a}(n^2-1)$ $\tau_n^e = 0$

The elastic stability of a structure is affected by any deformations, and a nonlinear theory is needed if this factor is to be included in the analysis. The present theory neglects the influence of prebuckling deformations on elastic stability, but nonlinear analysis using numerical methods indicates that the performance of the simplified theory is very satisfactory in the prebuckling region[14, 19]. The simplified theory has also been used elsewhere to predict the response of tubes under the influence of nonuniform hoop compression[19], and comparison with more complex theories[14, 20] indicates that the simplified theory is quite suitable for solving that more complex problem.

7. SUMMARY AND CONCLUSIONS

The stability of buried circular cylinders subjected to uniform prestress has been examined. The shell theories of Flugge[9] and Herrmann and Armenakas[10] were employed in the formulation of the buried tube stability problem. Three alternative load behaviours were used to modify the ground response in preference to the more complex ground model adopted by Forrestal and Herrmann[3] in their solution. The importance of defining the elastic ground parameters with respect to the ground in its prestressed state was also noted, even for perfectly elastic continua.

A simplified theory was developed using inextensional shell theory, which accurately reproduces the results of the more complex theories. It was used to obtain a general parametric solution for the critical hoop compression, in the form of influence and correction factors for the effects of relative cylinder stiffness, relative thickness, Poisson's ratio of the ground, interface condition, and loading behaviour. As a result, the following conclusions have been made:

(1) Loading behaviour, the manner in which the ground applies stresses to the cylinder as it deforms, can be neglected whenever the ground provides significant support to the cylinder (usually when the critical mode $n_{cr} \geqslant 6$).
(2) The nature of the interface does not affect the elastic stability of cylinders buried in incompressible soil (when Poisson's ratio is 1/2).
(3) In general, smooth cylinders receive less ground support than those which are rough, and the difference in critical hoop compression may be up to 15% when the Poisson's ratio of the ground is low.
(4) For thicker tubes, the shear tractions acting at the interface between the ground and structure induce bending moments about the shell midsurface which stabilise the system. It is conservative to neglect this effect.
(5) Use of the parametric solution leads to simple and accurate estimates of the critical hoop compression for a wide range of ground support conditions.

The simplified theory was extended to permit the estimation of prebuckling displacements and stress resultants for buried cylinders of imperfect shape, or when initial field stress conditions are nonhydrostatic. The authors have compared the simplified solution with more complex numerical and analytical solutions in another reference[19]. That comparison demonstrates that the simplified theory can be successfully used to solve more complex buried tube problems.

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